Recurrent Events and Renewal Theory

(based on W. Feller's *An Introduction to Probability Theory and Its Applications*, 3rd edition)

What is a recurrent event, $\mathcal{E}$?

Is any event that happens from time to time and with a certain probability a recurrent event?

→ Loosely speaking, an event may be considered a recurrent event if, after any occurrence of $\mathcal{E}$, subsequent trials form a probabilistic replica of the whole experiment. Trials of an experiment dealing with recurrent events "start from scratch" after each occurrence of $\mathcal{E}$.

The simplest example of a recurrent event may be when a Bernoulli process produces a "success".

$\longrightarrow$ What is a Bernoulli process?

By changing the definition of $\mathcal{E}$, we can discover many different types of recurrent events that can be generated by underlying Bernoulli trials.

→ $\mathcal{E}$ stands for "when the accumulated number of successes and failures are equal." Let $X_1, X_2, \ldots$ represent independent trial results, assuming the values 1 and -1 with probabilities p and q, respectively.

$$S_0 = 0 \qquad\qquad S_n = X_1 + X_2 + \ldots + X_n$$

$\mathcal{E}$ occurs iff $S_n = 0$.

→ How is $\mathcal{E}$ recurrent?

→ Anything peculiar about when $\mathcal{E}$ can happen?

→ Return to equilibrium through negative values

Similar to previous example, but $\mathcal{E}$ occurs iff:

$$S_n = \emptyset \qquad \text{and} \qquad S_1 < \emptyset,\ S_2 < \emptyset,\ \ldots,\ S_{n-1} < \emptyset$$

→ What if we want $\lambda > 1$ times as many successes as failures? Such events are termed "periodic" as $\mathcal{E}$ can only occur at trials $\lambda, 2\lambda, 3\lambda, \ldots n\lambda$.

$\longrightarrow$ Ladder Variables

Using previous notation, $\mathcal{E}$ happens at the nth trial if S_n exceeds all preceding sums:

$$S_n > \emptyset \quad \text{and} \quad S_n > S_{n-1},\ S_{n-1} > S_{n-2}, \ldots, S_1 > S_0$$

or

$$S_n > \emptyset \quad \text{and} \quad S_n > S_1,\ S_n > S_2, \ldots, S_n > S_{n-1}$$

$\longrightarrow$ Do these define the same $\mathcal{E}$?

→ Success Runs in Bernoulli trials

How can we define a run of length r in such a way to make it a recurrent event $\mathcal{E}$?

→ A run of at least length r?

→ A run of exactly length r?

→ A first run of length r?

Consider the sequence: SSSSFSSSSSSFF

Count the number of runs at different r values and note the trials where $\mathcal{E}$ takes place, starting at trial 1.

r	#	Where?
2		
3		

→ Simplest Queuing Process

Defined by a sequence of Bernoulli trials, e.g. (S, F, S, S, S, F, ...) and a sequence of random variables $X_1, X_2, \ldots$ assuming only positive integers. The X_k have a common distribution and are independent of each other and of the Bernoulli trials. A customer help desk, or "server", is either free or busy — initially it is free. A success at the n^{th} trial signifies the arrival of a customer at the server. If the server is free, the customer is helped immediately and for duration X_n. If the server is busy, the customer gets in line and waits.

→ What is the recurrent event $\mathcal{E}$ here?

→ Consider the following scenario

There is a deck of cards and my hand. Initially, my hand is empty. For every trial I draw at random a card from the deck and hold it in my hand.

→ When I get my first pair is that a recurrent event?

→ What if $\mathcal{E}$ is everytime I get a pair?

→ How might this process be modified to produce recurrent events?

→ Consider any recurrent pattern $\mathcal{E}$.
What can you say about the waiting times between successive occurrences of $\mathcal{E}$?

→ What is a geometric distribution? Can every recurrent event process be described as having a geometric dist?

→ What other process that this group discusses often is said to be "memory free."

→ Why is identifying a recurrent pattern possibly useful?

A recurrent pattern implies that certain probability distributions and other related random variables tend to definite limits as $n \longrightarrow \infty$. In Feller's words, the existence of a recurrent pattern enables us to prove the existence of a steady state and to analyze its dominant features.

→ Some Definitions and Relations

The crux of the subsequent formulations and proofs rests on the following two probability distributions

$$u_n = P\{\mathcal{E} \text{ occurs at the } n^{th} \text{ trial}\}$$

$$f_n = P\{\mathcal{E} \text{ occurs for the first time only at the } n^{th} \text{ trial}\}$$

For future convenience, we define

$$u_0 = 1 \qquad f_0 = \emptyset$$

$\longrightarrow$ Furthermore, we introduce the generating functions

$$F(s) = \sum_{k=1}^{\infty} f_k s^k \qquad U(s) = \sum_{k=1}^{\infty} u_k s^k$$

note: $\{u_k\}$ is not a probability distribution and in some cases

$$\sum_{k=1}^{\infty} u_k = \infty$$

However, the events "$\mathcal{E}$ occurs for the first time at the n^{th} trial are mutually exclusive, and therefore

$$f = F(1) = \sum_{n=1}^{\infty} f_n \leq 1$$

$\longrightarrow$ What does $1-f$ signify?

→ We can express the basic relation of

$$u_n = f_1 u_{n-1} + f_2 u_{n-2} + \dots + f_n u_0 \qquad n \geq 1$$

Recognizing the convolution $\{f_k\} * \{u_k\}$ on the right with the generating functions, and the sequence $\{u_n\}$ on the left with u_0 missing, we substitute and get $U(s) - 1 = F(s)U(s)$. With a little algebra, we produce the first main theorem:

$$U(s) = \frac{1}{1 - F(s)}$$

→ More Definitions

A recurrent event $\mathcal{E}$ will be called persistent if $f=1$ and transient if $f<1$. The probability f^r that a transient event $\mathcal{E}$ will happen at least r times tends to zero, but remains constant at 1 (by definition) for persistent events.

Persistent $\mathcal{E}$ occur infinitely often whereas transient $\mathcal{E}$ occur a finite number of times.

→ What can we say about a game of $r \geq 2$ coins? where $\mathcal{E}$ is when all r coins have expressed the same number of heads and without knowing the generating functions.

define $u_n =$

Through "magic" Feller turns that RHS into

$$u_n \sim \frac{1}{\sqrt{r}} \left(\frac{2}{\pi n} \right)^{\frac{1}{2}(r-1)}$$

and concludes that $\sum u_n$ diverges when $r \leq 3$ but converges when $r \geq 4$. What does this mean for $\mathcal{E}$?

→ How I used some of this insight

I wanted to see if my basketball player data was obeying recurrent event theory by analyzing the success runs of various lengths. Feller graciously supplied the following equations

$$\mu = \frac{1-p^r}{qp^r} \quad \text{and} \quad \sigma^2 = \frac{1}{(qp^r)^2} - \frac{2r+1}{qp^r} - \frac{p}{q^2}$$

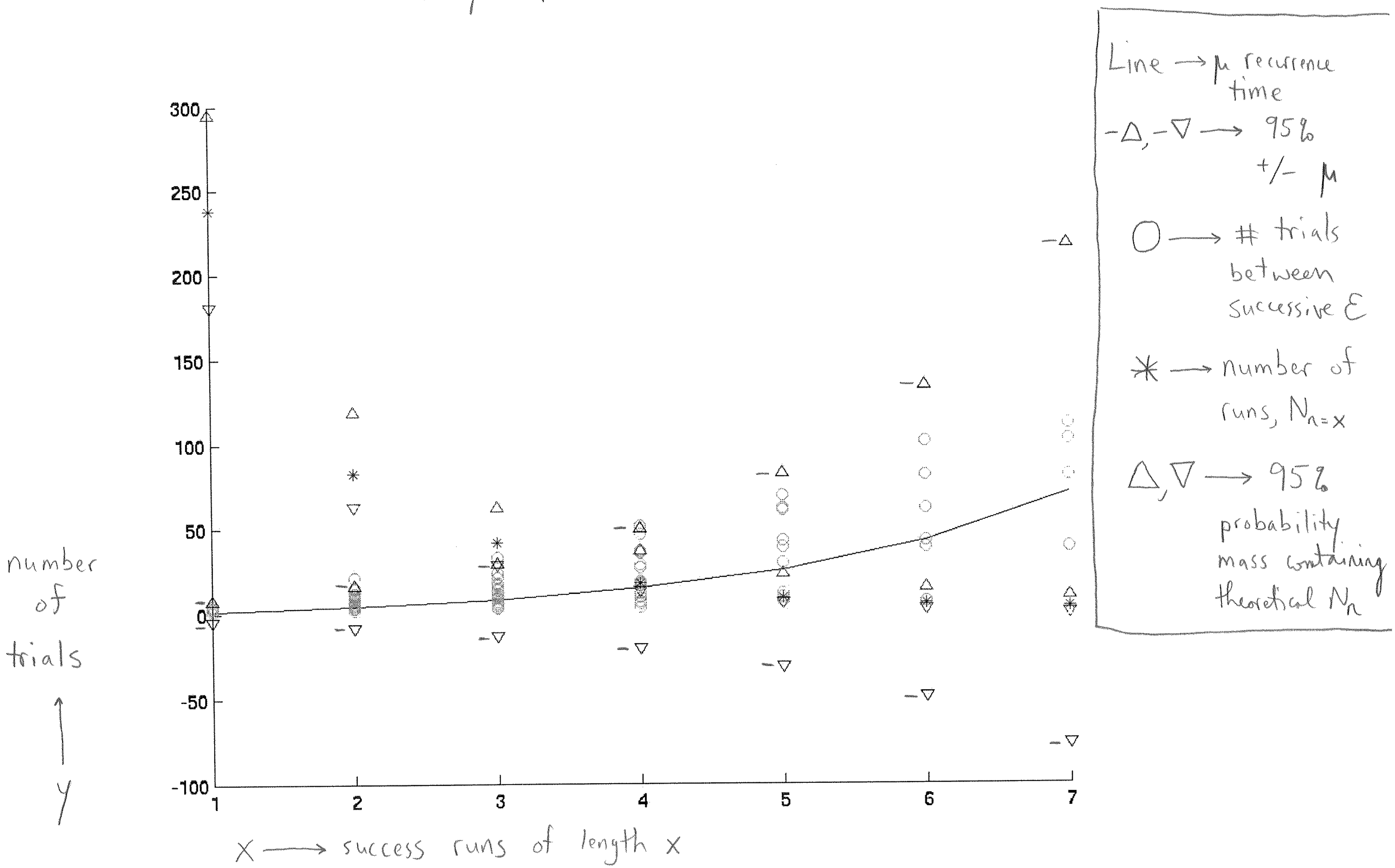
describing mean and variance for the recurrence times of length r runs.

→ Furthermore, he states for a large n, the number N_n of runs of length r produced in n trials is approximately normally distributed that is for fixed $\alpha < \beta$ the probability that

$$\frac{n}{\mu} + \alpha\sigma\sqrt{\frac{n}{\mu^3}} < N_n < \frac{n}{\mu} + \beta\sigma\sqrt{\frac{n}{\mu^3}}$$

tends to $\mathfrak{N}(\beta) - \mathfrak{N}(\alpha)$.

Does this player perform like a good recurrent event?



Darryl Dawkins made 238 of 383 shots. $p = .6214$

```
function out=MeanRunRecurTime(data, from, to)
% measuring time occurence of success runs only (not runs of missed shots)


% p is the bias
%data=~data;
%sum(data)
p=sum(data)/length(data);
q=1-p;      %good old q
Beta=.99;
Alpha=.01;
MyBeta=2;
MyAlpha=-2;

%
r=from:to;
N=zeros(1,length(r));

hold on;
mu = (1-p.^r)./( q.*p.^r);
plot(r,mu);

for j=from:to,                                          %streak length
    i=1;                                                %start from beginning
    laststreak=0;                                       %keeps track of time of last
    while i<length(data)
        streak=0;                                       %boolean, streak possible?
        if data(i)==1 && (i+j-1 <= length(data))
            streak=1;                                   %beginning of possible strea
            for t=i+1:i+j-1,
                if data(t)==0
                    streak=0;                           %kill streak if hit 0
                end
            end
            if streak==1                                %found streak
                N(j-from+1)=N(j-from+1)+1;              %increment number of j strea
                i=i+j-1;                                %move i to end of streak
                plot(j, i-laststreak, 'go');            %print position of streak
                laststreak=i;                           %becomes last streak
            end
        end
        i=i+1;                                          %advance i in any case
    end
end

variance=(1./(q.*p.^r).^2) - ((2.*r+1)./q.*p.^r) - p./q^2;
top=(length(data)./mu)+MyBeta.*sqrt(length(data).*variance./mu.^3);
bottom=(length(data)./mu)+MyAlpha.*sqrt(length(data).*variance./mu.^3);

plot(r,top,'r^',r,bottom,'rv', r,N(1,:), 'r*');

toptime=mu+2.*sqrt(variance);
bottomtime=mu-2.*sqrt(variance);

plot(r,toptime,'b^',r,bottomtime,'bv');

out=top;

hold off;
```